# Lowering operators for $O^+(9) \to O^+(3)$ and $O^+(7) \to O^+(3)$



1974 J. Phys. A: Math. Nucl. Gen. 7 1497

(http://iopscience.iop.org/0301-0015/7/13/003)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 13, 1974. Printed in Great Britain. © 1974

# Lowering operators for $O^+(9) \rightarrow O^+(3)$ and $O^+(7) \rightarrow O^+(3)$

L S R K Prasad

Department of Applied Mathematics, Andhra University, Waltair–530003, India

Received 5 December 1973, in final form 19 March 1974

**Abstract.** Lowering operators for $O^+(9) \rightarrow O^+(3)$ and $O^+(7) \rightarrow O^+(3)$ are explicitly constructed. This solves the problem of obtaining an $O^+(3)$ basis (non-orthogonal) for irreducible representations of U(9) and U(7).

## 1. Introduction

In the application of group theoretical techniques to physical problems, like the spectroscopy of a single $l$ shell, one has to deal with the group chain

$$\mathrm{SU}(2l+1) \supset \mathrm{O}^+(2l+1) \supset \mathrm{O}^+(3).$$

Prasad (1972) has obtained explicitly a polynomial basis for the irreducible representations (IR) of $U(n)$ in the chain $U(n) \supset O^+(n)$. But to be of physical significance one has to give an $O^+(3)$ basis for an IR of $U(n)$. Hughes (1973) has given operators useful for obtaining an orthogonal basis for $SU(3) \supset O^+(3)$. Flores *et al* (1965) gave the lowering operators for $O^+(5) \rightarrow O^+(3)$ and $SU(3) \rightarrow O^+(3)$, and their basis is a non-orthogonal one. Adopting the method of Flores *et al* (1965) and using the fact that the generators of any semi-simple Lie group can be expressed (Stone 1961) as the generators of a subgroup together with irreducible tensor operators under this subgroup, we explicitly construct here lowering operators for $O^+(9) \rightarrow O^+(3)$ and $O^+(7) \rightarrow O^+(3)$, as these are of immediate physical interest. This solves the problem of obtaining an $O^+(3)$ basis (non-orthogonal) for IR of U(9) and U(7). The earlier notation of Prasad (1972) is adopted.

## 2. Construction of the lowering operators

Let us denote the highest weight polynomial (HWP) of the IR $[\lambda] \equiv [\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ of $O^+(9)$ contained in an IR $\{h\} \equiv \{h_1, h_2, \ldots, h_8\}$ of U(9) as $P^{\{h\}[\lambda]}$, which can be obtained by the method of Prasad (1972). Now our aim is to obtain the HWP of the various IR, $L$, of $O^+(3)$ contained in the IR $[\lambda]$ of $O^+(9)$.

The following linear combinations:

$$H = \sum_{i=1}^{l} (l-i+1)\Lambda_i^i,$$

$$E_+ = \frac{1}{\sqrt{2}} \sum_{m=0}^{l-1} [(l+m+1)(l-m)]^{1/2} \Lambda_{l-m}^{l+1-m},$$

and

$$E_{-} = \frac{1}{\sqrt{2}} \sum_{m=0}^{l-1} [(l+m+1)(l-m)]^{1/2} \Lambda^{l-m}_{l+1-m}, \tag{1}$$

where $\Lambda^{\mu'}_{\mu}$ are the generators of $O^+(9)$, with $l = 4$, are the generators of $O^+(3) \subset O^+(9)$. With respect to this $O^+(3)$ subgroup, the generators of $O^+(9)$ can be combined to form the components of irreducible tensors of ranks $k = 7$, 5 and 3. From the commutation relations of $H$ and $E_+$ with the generators of $O^+(9)$, one can easily see that

$$T^{(7)}_7 = \Lambda^8_1, \qquad T^{(5)}_5 = 2\Lambda^6_1 - 3\Lambda^7_2$$

and

$$T^{(3)}_3 = -\frac{\sqrt{7}}{5}\Lambda^4_1 - \frac{\sqrt{14}}{2\sqrt{5}}\Lambda^5_2 + \Lambda^6_3.$$

The other components of these tensors can be obtained by using equations (152) of Racah (1951). From (1) it follows that the HWP of the IR $[\lambda]$ of $O^+(9)$, $P^{\{h\}[\lambda]}$, is also the HWP of the IR $L = 4\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4$ of $O^+(3)$, which is the highest IR of $O^+(3)$ contained in the IR $[\lambda]$ of $O^+(9)$. We therefore denote this HWP as $P^{\{h\}[\lambda]}_L$. Now we construct certain lowering operators which, when operating on $P^{\{h\}[\lambda]}_L$, give us the HWP of the various other IR of $O^+(3)$ contained in the same IR, $[\lambda]$, of $O^+(9)$. We denote the lowering operator, which operating on the HWP of an IR, $L$, of $O^+(3)$ gives us the HWP of the IR, $L'$ of $O^+(3)$, as $\mathscr{L}^{(k)}_{LL'}$. The meaning of $k$ will become clear later. The lowering operator $\mathscr{L}^{(k)}_{LL'}$, which must be a function of the generators of $O^+(9)$, must therefore satisfy the following conditions:

$$H\mathscr{L}^{(k)}_{LL'}P^{\{h\}[\lambda]}_L = L'\mathscr{L}^{(k)}_{LL'}P^{\{h\}[\lambda]}_L, \tag{2}$$

and

$$E_+\mathscr{L}^{(k)}_{LL'}P^{\{h\}[\lambda]}_L = 0. \tag{3}$$

For $\mathscr{L}^{(k)}_{LL'}$ to satisfy the condition (2), as a simple function, we choose it as

$$\mathscr{L}^{(k)}_{LL'} = \sum_{q=k}^{-k} A_q (E_-)^{q+L-L'} T^{(k)}_q, \tag{4}$$

where $A_q$ are constants, to be determined. The $k$ appearing in the lowering operator symbol indicates that we are using irreducible tensors of rank $k$ in the right-hand side of (4). Terms corresponding to negative values of $q+L-L'$, for a given $L$ and $L'$, are to be dropped in the summation on the right-hand side of (4). As we have already obtained irreducible tensors of ranks $k = 7$, 5 and 3, out of the generators of $O^+(9)$, substituting them separately in the right-hand side of (4) we get three lowering operators $\mathscr{L}^{(7)}_{LL'}$, $\mathscr{L}^{(5)}_{LL'}$ and $\mathscr{L}^{(3)}_{LL'}$.

In the following, we will explain how the constants $A_q$, $q = 3, \ldots, -3$, occurring in the expression for $\mathscr{L}^{(3)}_{LL'}$ are determined. It can be easily seen that

$$[\mathscr{L}^{(3)}_{LL'}, H] = (L-L')\mathscr{L}^{(3)}_{LL'}.$$

Thus condition (2) is satisfied by $\mathscr{L}^{(3)}_{LL'}$. We determine the constants $A_q$ such that condition (3) is also satisfied. We note that if

$$[E_+, \mathscr{L}^{(3)}_{LL'}]P^{\{h\}[\lambda]}_L = 0$$

then

$$E_+\mathscr{L}^{(3)}_{LL'}P^{\{h\}[\lambda]}_L = \mathscr{L}^{(3)}_{LL'}E_+P^{\{h\}[\lambda]}_L = 0,$$

that is, condition (3) is satisfied. Now $[E_+, \mathscr{L}^{(3)}_{LL'}]P^{\{h\}[\lambda]}_L = 0$ gives us equations of the form

$$x_1A_3 + x_2A_2 = 0, \qquad x_3A_2 + x_4A_1 = 0, \qquad \ldots$$
$$x_{11}A_{-2} + x_{12}A_{-3} = 0 \qquad \text{and} \qquad x_{13}A_{-3} = 0.$$

From the first six equations we determine $A_2, A_1, \ldots, A_{-3}$ in terms of $A_3$. If we substitute this value of $A_{-3}$ in the last equation we get

$$(3+L-L')(2+L-L')(1+L-L')(L-L')(L-L'-1)(L-L'-2)(L-L'-3)$$
$$\times(4+L+L')(3+L+L')(2+L+L')(1+L+L')(L+L')$$
$$\times(L+L'-1)(L+L'-2)A_3 = 0.$$

As we are interested in $L' < L$, this equation is satisfied whenever $L' = L-1$ or $L-2$ or $L-3$. Therefore choosing $A_3$ as $-(6!)^{1/2}$ we can obtain the values of $A_2, A_1, \ldots, A_{-3}$, and the $\mathscr{L}^{(3)}_{LL'}$, with these constants incorporated, satisfies the equation

$$[E_+, \mathscr{L}^{(3)}_{LL'}]P^{\{h\}[\lambda]}_L = 0,$$

which in turn implies that the $\mathscr{L}^{(3)}_{LL'}$ satisfies the condition (3), whenever $L' = L-1$ or $L-2$ or $L-3$.

In a similar way, we construct $\mathscr{L}^{(7)}_{LL'}$, which can be used to reduce an IR, $L$, to an IR $L' = L-1, L-2, \ldots, L-7$, and $\mathscr{L}^{(5)}_{LL'}$, which can be used to reduce an IR, $L$, to an IR, $L' = L-1, L-2, \ldots, L-5$, of $O^+(3)$. Thus, we get the final form for the lowering operators, $\mathscr{L}^{(k)}_{LL'}$, as

$$\mathscr{L}^{(k)}_{LL'} = \sum_{q=k}^{-k} (-1)^q \left(\frac{(k+q)!}{2^{k-q}(k-q)!}\right)^{1/2} \frac{(L-L'+k)!(L+L'+k+1)!}{(L-L'+q)!(L+L'+q+1)!}(E_-)^{q+L-L'}T^{(k)}_q, \tag{5}$$

where $k = 7, 5, 3$.

As we have chosen $A_3$ in a particular way, the lowering operator, $\mathscr{L}^{(3)}_{LL'}$, is unique except for a multiplicative constant and this is enough for our present purpose, since the HWP of an IR is unique except for a multiplicative constant. A similar argument holds for $\mathscr{L}^{(7)}_{LL'}$, and $\mathscr{L}^{(5)}_{LL'}$. It can be easily seen that

$$C\mathscr{L}^{(k)}_{LL'}P^{\{h\}[\lambda]}_L = L'(L'+1)\mathscr{L}^{(k)}_{LL'}P^{\{h\}[\lambda]}_L, \qquad k = 7, 5, 3,$$

where $C = H^2 - H + 2E_+E_-$ is the Casimir operator for $O^+(3)$.

From the HWP, $P^{\{h\}[\lambda]}_L$, using suitable products of the lowering operators $\mathscr{L}^{(k)}_{LL'}$, HWP of the various other IR of $O^+(3)$ contained in the IR $[\lambda]$ of $O^+(9)$ can be obtained. Linear combinations of $\mathscr{L}^{(k)}_{LL'}$ will not be helpful for us in this context. On the other hand we require products of $\mathscr{L}^{(k)}_{LL'}$ only. This is illustrated through the following example. The HWP of the IR [22] of $O^+(9)$ contained in the IR {22} of U(9) is $(12)^2$, in the notation of Prasad (1972). It is also the HWP of the highest IR, $L = 14$, of $O^+(3)$ contained in the IR [22] of $O^+(9)$. Let us write $P = (12)^2$. The IR [22] of $O^+(9)$ contains (Prasad *et al* 1974) the IR 14, $12^2$, 11, $10^3$, $9^2$, $8^4$, $7^3$, $6^5$, $5^3$, $4^5$, $3^2$, $2^4$, $0^2$ of $O^+(3)$.

$$\mathscr{L}^{(7)}_{14,12}P = 7(12)(23) - 10\sqrt{7}(13)^2 + 42(12)(14),$$
$$\mathscr{L}^{(5)}_{14,12}P = 40(12)(23) - 7\sqrt{7}(13)^2 + 6(12)(14),$$

and

$$\mathscr{L}^{(3)}_{14,12}P = -29(12)(23) - \sqrt{7}(13)^2 + 24(12)(14).$$

Out of these three HWP, only two are linearly independent and let us take $\mathscr{L}^{(7)}_{14,12}P = Q$ and $\mathscr{L}^{(3)}_{14,12}P = R$ as the HWP corresponding to the double occurrence of the IR 12 of $O^+(3)$. The HWP of the IR 11 of $O^+(3)$ can be obtained either as $\mathscr{L}^{(7)}_{14,11}P$ or $\mathscr{L}^{(5)}_{14,11}P$ or $\mathscr{L}^{(3)}_{14,11}P$ or $\mathscr{L}^{(7)}_{12,11}Q = \mathscr{L}^{(7)}_{12,11}\mathscr{L}^{(7)}_{14,12}P$ or $\mathscr{L}^{(7)}_{12,11}R = \mathscr{L}^{(7)}_{12,11}\mathscr{L}^{(3)}_{14,12}P$ or $\mathscr{L}^{(5)}_{12,11}\mathscr{L}^{(7)}_{14,12}P$ or $\mathscr{L}^{(5)}_{12,11}\mathscr{L}^{(3)}_{14,12}P$ or $\mathscr{L}^{(3)}_{12,11}\mathscr{L}^{(7)}_{14,12}P$ or $\mathscr{L}^{(3)}_{12,11}\mathscr{L}^{(3)}_{14,12}P$. Again, all these HWP of the IR 11 of $O^+(3)$ can be expressed in terms of any one of them, for example, $\mathscr{L}^{(7)}_{14,11}P$. This can be taken as the HWP corresponding to the single occurrence of the IR 11 of $O^+(3)$.

Once we get the HWP of an IR of $O^+(3)$, we can generate the whole basis by applying suitable powers of $E_-$ on it.

Proceeding similarly we construct the lowering operators for $O^+(7) \to O^+(3)$. The combinations (1), where $\Lambda^{\mu'}_{\mu}$ are the generators of $O^+(7)$, with $l = 3$, are the generators of $O^+(3) \subset O^+(7)$. With respect to this $O^+(3)$ subgroup, the generators of $O^+(7)$ can be combined as follows to obtain the components of irreducible tensors of ranks $k = 5$ and 3:

$$T^{(5)}_5 = \Lambda^6_1, \qquad T^{(3)}_3 = \Lambda^4_1 - \sqrt{2}\Lambda^5_2. \tag{6}$$

The other components of these tensors can be obtained by using equations (152) of Racah (1951). Using these two irreducible tensors, (6), and the $E_-$ of $O^+(3) \subset O^+(7)$ in (5) we get the two lowering operators $\mathscr{L}^{(5)}_{LL'}$ and $\mathscr{L}^{(3)}_{LL'}$ for $O^+(7) \to O^+(3)$.

## 3. Discussion

The combinations (1), where $\Lambda^{\mu'}_{\mu}$ are the generators of $O^+(2l+1)$, are the generators of $O^+(3) \subset O^+(2l+1)$. We can show that with respect to this $O^+(3)$ subgroup, the generators of $O^+(2l+1)$ can be combined to form the components of irreducible tensors of ranks $k = 2l-1,\ 2l-3, \ldots, 3$. Thus, the generalization to $O^+(2l+1) \to O^+(3)$ is evident. Though the lowering operators we have constructed here are found to be completely sufficient as far as the $O^+(3)$ contents of $O^+(9)$ and $O^+(7)$ (Prasad *et al* 1974, Hamermesh 1962) even when there is multiplicity for the IR of $O^+(3)$, we are at present not in a position to answer the multiplicity question in general. The lowering operators for $O^+(2l+1) \to O^+(3)$ and the answer to the multiplicity question we hope to give in a future paper. One can construct raising operators in a similar way, which starting from the lowest IR of $O^+(3)$ contained in a given IR of $O^+(9)$ will take us to the higher IR of $O^+(3)$ contained in the same IR of $O^+(9)$. In the case of lowering operators we know the HWP of the highest IR of $O^+(3)$ contained in a given IR of $O^+(9)$ directly by the method of Prasad (1972), whereas in the case of raising operators we do not know the HWP of the lowest IR of $O^+(3)$ contained in the given IR of $O^+(9)$.

## Acknowledgments

The author's thanks are due to Professor T Venkatarayudu and Professor T S G Krishnamurthy for many useful discussions. The author is grateful to the two referees for their valuable comments.

*Note added in proof.* Similar work has been carried out by the author and M Kondala Rao for the symplectic group and lowering operators for $\mathrm{Sp}(n) \to \mathrm{R}(3)$, $n = 6, 8, 10$, are explicitly constructed.

## References

Flores J, Chacon E, Mello P A and de Llano M 1965 *Nucl. Phys.* **72** 352–78

Hamermesh M 1962 *Group Theory and its Application to Physical Problems* (Reading, Massachusetts: Addison Wesley) chap 11

Hughes J W B 1973 *J. Phys.* A: *Math., Nucl. Gen.* **6** 48–58, 281–98

Prasad L S R K 1972 *Nucl. Phys.* B **36** 605–14

Prasad L S R K, Kondala Rao M and Madhusudana Rao P V 1974 *J. Phys.* A: *Math., Nucl. Gen.* **7** 449–51

Racah G 1951 *Group Theory and Spectroscopy* (Princeton, New Jersey: Institute for Advanced Study)

Stone A P 1961 *Proc. Camb. Phil. Soc.* **57** 460–8